

DYNAMICAL PROPERTIES OF THE ABSOLUTE PERIOD FOLIATION

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ABSTRACT. We show that the absolute period foliation of the principal stratum of abelian differentials on a surface of genus $g \geq 3$ is ergodic. We also investigate the absolute period foliation on affine invariant manifolds.

1. INTRODUCTION

The moduli space of abelian differentials on a surface of genus $g \geq 2$ naturally decomposes into *strata* of differentials with prescribed numbers of zeros. Period coordinates on such a stratum \mathcal{Q} are defined by evaluation of an abelian differential on a basis for relative homology. If \mathcal{Q} is a stratum of differentials with more than one zero then it admits a natural foliation whose leaves consist of differentials with (locally) fixed absolute periods. This foliation is smooth and has been analyzed in [McM13, McM14, MinW14]; it is called the *absolute period foliation*.

A smooth foliation of an orbifold \mathcal{Q} is called *ergodic* if any Borel subset of \mathcal{Q} which is saturated for the foliation either has full or vanishing Lebesgue measure. In [McM14], tools from homogeneous dynamics are used to show that the absolute period foliation of the principal stratum in $g = 2, 3$ is ergodic. Calsamiglia, Deroin and Francaviglia [CDF15] completely classified the closures of the leaves of the absolute period foliation. As a consequence, they obtain ergodicity of the absolute period foliation of the principal stratum in every genus.

Our main goal is to give a simple proof of the latter fact.

Theorem 1. *The absolute period foliation of the principal stratum is ergodic in every genus $g \geq 2$.*

We do not know whether the absolute period foliation of a stratum which is not principal is ergodic. Using an argument of Coudéne [C09], it is not hard to see that ergodicity is equivalent to the existence of a dense leaf.

By the groundbreaking work of Eskin, Mirzakhani and Mohammadi [EMM13], the closure of an orbit for the $SL(2, \mathbb{R})$ -action on any stratum \mathcal{Q} is an affine invariant manifold. Examples of non-trivial orbit closures are arithmetic Teichmüller

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curves. They arise from branched covers of the torus, and they are dense in any stratum of abelian differentials. Other examples of orbit closures different from entire components of strata can be constructed using more general branched coverings.

In Section 3 we introduce rigid and flexible tangent fields of the absolute period foliation and use this to investigate the principal boundary of an affine invariant manifold.

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2. THE ABSOLUTE PERIOD FOLIATION

Let \mathcal{Q} be a component of a stratum with $k \geq 2$ zeros in the moduli space of abelian differentials on a surface of genus $g \geq 2$. The absolute periods of an abelian differential $\omega \in \mathcal{Q}$ define a local submersion of orbifolds

$$\mathcal{Q} \supset U \rightarrow H^1(X, \mathbb{C})/\text{Aut}(X)$$

whose fibres are the intersections with \mathcal{Q} of the leaves of the *absolute period foliation* $\mathcal{AP}(\mathcal{Q})$. This foliation is transverse to the fibres of the canonical projection $\pi : \mathcal{Q} \rightarrow \mathcal{M}_g$ (here \mathcal{M}_g denotes the moduli space of Riemann surfaces of genus g) and to the orbits of the natural action of $SL(2, \mathbb{R})$.

The leaf $\mathcal{AP}(\omega)$ of $\mathcal{AP}(\mathcal{Q})$ through ω is locally a flat submanifold of \mathcal{Q} which can explicitly be described [MinW14, McM13].

Assume for the moment that \mathcal{Q} is the principal stratum. Denote by $Z(\omega)$ the zero set of $\omega \in \mathcal{Q}$. The cardinality of $Z(\omega)$ equals $2g - 2$. At each zero $p \in Z(\omega)$ there is an infinitesimal deformation of ω called the *Schiffer variation* [McM13] which is defined as follows.

Let X be the Riemann surface underlying ω . Choose a complex coordinate z for X near p so that in this coordinate, ω can be written as $\omega = (z/2)dz$. Such a coordinate is unique up to multiplication with -1 . Choose a vertical arc $A_t = i[-2u, 2u]$ in this coordinate where $t = u^2$. Slit S open along A_t and fold each of the two resulting arcs so that z is identified with $-z$ (see p.1235 of [McM13]).

The result is a new Riemann surface X_t with a distinguished horizontal arc B_t and a natural holomorphic map $f_t : X - A_t \rightarrow X - B_t$. The one-form ω_t with $f_t^* \omega_t = \omega$ is globally defined, and it only depends on the parameter t and on the choice of the zero p of ω . The Schiffer variation of X is

$$\text{Sch}(\omega, p) = dX_t/dt|_{t=0}.$$

It will be useful to have a geometric description of the deformation of the one-forms ω_t defining the Schiffer variation. Namely, there are four horizontal separatrices at p for the flat metric defined by ω . In a complex coordinate z near p so that $\omega = (z/2)dz$, the horizontal separatrices are the four rays contained in the real or the imaginary axis. The restriction of ω to these rays defines an orientation on the rays. With respect to this orientation, the two rays contained in the real axis are outgoing from p , while the rays contained in the imaginary axis are incoming.

The Schiffer variation slides the singular point backwards along the incoming rays in the imaginary axis. Thus if one of the two separatrices in the imaginary axis is a saddle connection for ω , then the flat length of the corresponding saddle connection for ω_t is decreasing with t .

If ω has a zero of order $n \geq 2$ at p then the Schiffer variation at p is defined as follows (see p.1235 of [McM13]). Choose a coordinate z near p so that $\omega = z^n dz$ in this coordinate. This choice of coordinate is unique up to multiplication with $e^{\ell 2\pi i / (n+1)}$ for some $\ell \leq n$. There are $n+1$ horizontal separatrices at p for the flat metric defined by ω whose orientations point towards p . For small $u > 0$ cut the surface S open along the initial subsegments of length $2u$ of these $n+1$ horizontal segments. The result is a $2n+2$ -gon which we refold as in the case of a simple zero.

Now let C be a smooth simple loop enclosing the zero $p \in Z(\omega)$. Then the Schiffer variation at p is the real part $\delta(C, -1/\omega)$ of a complex twist deformation $\delta(C, v)$ of X about C where v is a holomorphic vector field along C .

Namely, there is a vector $\text{tw}(C)$ tangent to $\mathcal{AP}(\mathcal{Q})$ at ω defined by

$$\langle \text{tw}(C), E \rangle = C \cdot E$$

where $E \in H_1(X, Z(\omega))$ and where \cdot is the natural intersection pairing

$$H_1(X - Z(\omega)) \times H_1(X, Z(\omega)) \rightarrow \mathbb{Z}.$$

The tangent space at ω to the absolute period foliation is generated by the transformations $\text{tw}(C_p)$, $p \in Z(\omega)$, subject to the relation $\sum \text{tw}(C_{p_i}) = 0$ (see p.1236 of [McM13]). If \mathcal{Q} is the principal stratum then the leaves of the absolute period foliation $\mathcal{AP}(\mathcal{Q})$ have complex dimension $2g-3$. We refer to [McM13] for details and an explanation of these notations.

Note that the tangent bundle of $\mathcal{AP}(\mathcal{Q})$ is naturally equipped with a complex structure J as well as with a real structure. The subbundle of $T\mathcal{AP}(\mathcal{Q})$ spanned by the twist deformations corresponding to the Schiffer variations is a maximal real subbundle for this real structure. By abuse of notation, we call a twist deformation corresponding to a Schiffer variation again a Schiffer variation, i.e. we view Schiffer variations as tangent vectors of the absolute period foliation.

Example 2.1. Let ω_1, ω_2 be two abelian differentials on two closed surfaces S_1, S_2 of genus g_1, g_2 . Assume that the area of ω_i is a_i for some $a_i > 0$ with $a_1 + a_2 = 1$. Cut a small horizontal slit into S_1, S_2 of the same length. The differentials ω_i define an orientation of these slits. Glue S_1 to S_2 with an orientation reversing isometry along the slits. The result is an area one abelian differential ω on a surface of genus $g_1 + g_2$ with two singular points p_1, p_2 which are connected by two homologous horizontal saddle connections of the same length. Assume that the orientation defined by ω of these saddle connections points from p_1 to p_2 . The deformation induced by the Schiffer variation corresponding to the parameters $(-1, 1)$ decreases the length of the slit and limits in a surface with nodes. This surface with nodes consists of the surfaces S_1, S_2 attached at one point, equipped with an abelian differential which maps to the differentials ω_1, ω_2 by the marked point forgetful map.

Let again \mathcal{Q} be a component of a stratum with $k \geq 2$ zeros and let $\hat{\mathcal{Q}}$ be a finite normal cover of \mathcal{Q} such that there is a consistent numbering of the zeros of $q \in \hat{\mathcal{Q}}$ varying continuously with q . We may assume that \mathcal{Q} is the quotient of $\hat{\mathcal{Q}}$ by the action of a subgroup of the symmetric group in k variables.

Let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ be any k -tuple of *real* numbers with $\sum_i a_i = 0$. Then \mathbf{a} defines a smooth vector field $X_{\mathbf{a}}$ on $\hat{\mathcal{Q}}$ as follows. For each $\omega \in \hat{\mathcal{Q}}$, the value of $X_{\mathbf{a}}$ at ω is the Schiffer variation for the parameters (a_1, \dots, a_k) at the numbered zeros of ω . Thus $X_{\mathbf{a}}$ is tangent to the absolute period foliation.

The *Teichmüller flow* Φ^t on \mathcal{Q} lifts to a smooth flow on $\hat{\mathcal{Q}}$ denoted by the same symbol. Its derivative acts on the tangent bundle of $\hat{\mathcal{Q}}$. We have

Lemma 2.2. $d\Phi^t X_{\mathbf{a}} = e^t X_{\mathbf{a}}$.

Proof. Let $\omega \in \hat{\mathcal{Q}}$; then the horizontal foliation \mathcal{F} of ω is defined by $\omega(T\mathcal{F}) \in \mathbb{R}$. The Teichmüller flow expands the horizontal foliation \mathcal{F} of ω with the expansion rate $e^{t/2}$. Thus if A_s is an arc of length $4\sqrt{s}$ in the imaginary axis for the preferred coordinate near p (recall that this arc is horizontal for the flat metric defined by ω) then the image of A_s in $\Phi^t \omega$ is an arc of length $4e^{t/2}\sqrt{s} = 4\sqrt{e^t s}$ in the imaginary axis of a preferred coordinate. Taking derivatives shows the claim. \square

The vector field $X_{\mathbf{a}}$ defines a flow $\Lambda_{\mathbf{a}}^t$ on $\hat{\mathcal{Q}}$. This flow is incomplete as a horizontal saddle connection may give rise to a finite flow line limiting on a lower dimensional stratum (see p.1235 of [McM13]). There may also be limit points on surfaces with nodes as described in Example 2.1. However, if q does not have any horizontal saddle connection then the flow line of $\Lambda_{\mathbf{a}}^t$ through q is defined for all times [MinW14]. As almost every point with respect to the Lebesgue measure λ is a differential without horizontal saddle connection, $\Lambda_{\mathbf{a}}^t$ defines a flow on a subset of $\hat{\mathcal{Q}}$ of full Lebesgue measure.

For $\mathbf{a} \neq \mathbf{b}$ the flows $\Lambda_{\mathbf{a}}^t$ and $\Lambda_{\mathbf{b}}^s$ commute and hence these flows fit together to a (local) action of the group \mathbb{R}^{k-1} on $\hat{\mathcal{Q}}$. This action is smooth, and its local orbits naturally develop to a smooth foliation of $\hat{\mathcal{Q}}$ called the *real REL foliation* [MinW14]. This foliation is a subfoliation of the absolute period foliation.

A leaf of the *local strong unstable foliation* of $\hat{\mathcal{Q}}$ consists of abelian differentials with the same horizontal foliation (up to Whitehead moves).

Lemma 2.3. *The real REL foliation is a subfoliation of the strong unstable foliation of $\hat{\mathcal{Q}}$ which is invariant under the action of the Teichmüller flow and under holonomy along the strong stable foliation.*

Proof. By construction, the Schiffer variation defined by the vector field $X_{\mathbf{a}}$ preserves the horizontal foliation of an abelian differential up to Whitehead moves. Hence the vector fields $X_{\mathbf{a}}$ are tangent to the strong unstable foliation of $\hat{\mathcal{Q}}$. As a consequence, the real REL foliation is a subfoliation of the strong unstable foliation, and it is smooth. We refer to [McM14] for a detailed analysis of this foliation in the case $g = 2$.

Together with Lemma 2.2, this implies invariance under the action of the Teichmüller flow. Invariance under holonomy along the strong stable foliation follows from the fact that a leaf of the real REL foliation can be characterized as the set of all abelian differentials in the stratum with fixed horizontal foliation and the property that the vertical foliations all define the same absolute cohomology class (see [MinW14]). The lemma follows. \square

Remark 2.4. It is easy to see that the flows $\Lambda_{\mathbf{a}}^t$ preserve the Lebesgue measure λ of the stratum. It is an interesting question whether any of these flows is ergodic. Our proof of ergodicity of the absolute period foliation does not give any information to this end.

The above discussion shows that the absolute period foliation has an affine structure (see [MinW14] and p.1236 of [McM13] for more details and compare also [LNW15]).

Recall that there is a natural circle action on \mathcal{Q} . To a point e^{is} on the unit circle $S^1 \subset \mathbb{C}^*$ and a quadratic differential q we associate the differential $e^{is}q$. For $\mathbf{a} \in \mathbb{R}^k$ with zero mean, for $e^{is} \in S^1$ and for $\omega \in \mathcal{Q}$ let

$$\Lambda_{e^{is}\mathbf{a}}^t(\omega) = e^{-is}\Lambda_{\mathbf{a}}^t(e^{is}\omega).$$

Then $(t, \omega) \rightarrow \Lambda_{e^{is}\mathbf{a}}^t\omega$ defines a flow on $\hat{\mathcal{Q}}$ which preserves the absolute period foliation.

Following [EMZ03] we define the *principal boundary* of the component \mathcal{Q} of a stratum as follows. Let $\omega \in \mathcal{Q}$ and assume that ω has a horizontal saddle connection and that the set of horizontal saddle connections of ω is a forest, i.e. it does not have cycles. Let p_1, p_2 be the endpoints of such a saddle connection α , chosen such that α points from p_1 to p_2 with respect to the orientation defined by ω , and let \mathbf{a} be the vector $(-1, 1)$ at p_1, p_2 . (Strictly speaking, this is only defined in $\hat{\mathcal{Q}}$, but the choice of α singles out two zeros of ω and hence this makes also sense in \mathcal{Q}). Then the arc $t \rightarrow \Lambda_{\mathbf{a}}^t\omega$ limits on a differential ζ for which the points p_1, p_2 coalesce, and there are no other identifications of zeros. The differential ζ is contained in a component of a stratum in the boundary of \mathcal{Q} , and we call such a component a *finite core face* of the principal boundary of \mathcal{Q} . The dimension of a finite core face of \mathcal{Q} equals $\dim(\mathcal{Q}) - 1$.

A second degeneration which gives rise to a point in the principal boundary is the contraction of two or more homologous saddle connections. In this case the resulting surface is a surface consisting of two or more smooth components which are connected at nodes. The sum of the genera of these surfaces equals g , and the resulting abelian differential has a regular point or a zero at a node. We call a component of abelian differentials on surfaces with nodes arising in this way an *infinite face*. We call the infinite face a *core face* if it consists of surfaces comprised of two components which are attached at a single separating node. Note that there are up to $\lfloor g/2 \rfloor$ core faces which correspond to the *type* of the decomposition, i.e. to a decomposition $g = g_1 + g_2$ with $g_1, g_2 \geq 1$. The node defines a marked point on each of the components of the surface with nodes. We call a point in a core face a *regular point* if the node is not a zero of the abelian differential on a component

surface. A point which is not regular is called *singular*. The set of singular points is of codimension one.

To summarize, there is a decomposition of the principal boundary of \mathcal{Q} into *faces* (see p.76 of [EMZ03]). Each face either is a component of a stratum in the adherence of \mathcal{Q} with fewer zeros, or it corresponds to a *configuration* which consists of a decomposition of the surface into surfaces S_i of genus g_i with $\sum_i g_i = g$, a combinatorial configuration of attachment data which organizes the glueing at the nodes and numbers $a_j \geq 0$ which describe the order of the zero of the differentials at the node (see [EMZ03] for more details). If we denote by $\overline{\mathcal{Q}}$ the union of \mathcal{Q} with its principal boundary, then the core faces are the faces of codimension one in $\overline{\mathcal{Q}}$.

For the remainder of this section we assume that \mathcal{Q} is the principal stratum. The following structure theorem is Lemma 9.8 of [EMZ03]. For its formulation, note that $\overline{\mathcal{Q}}$ properly contains the entire moduli space of area one abelian differentials. For $\epsilon > 0$ let $B(\epsilon)$ be the disk of radius ϵ in the complex plane.

Proposition 2.5. *Let \mathcal{F} be an infinite core face of the principal boundary and let $\omega \in \mathcal{F}$ be a regular point. Then there is a number $\epsilon > 0$, and there is a neighborhood V of ω in \mathcal{F} , a neighborhood U of ω in $\overline{\mathcal{Q}}$, and a homeomorphism $\varphi : V \times B(\epsilon) \rightarrow U$ with the following properties.*

- (1) $\varphi(x, 0) = x$ for all $x \in V$.
- (2) $\varphi(\{x\} \times B(\epsilon)) \subset \mathcal{AP}(x)$.

Proof. A regular point $z \in \mathcal{F}$ is defined by two abelian differentials ω_1, ω_2 on surfaces S_1, S_2 attached at a marked point p_1, p_2 . The marked point p_i is a regular point for ω_i . Take a vector γ in the complex plane of sufficiently small length $r < \epsilon$, slit S_1, S_2 open at the marked points in direction of γ and glue the abelian differentials ω_1, ω_2 along the slits. The resulting differential $\varphi(z, \gamma)$ has the same absolute periods as z .

By Lemma 9.8 of [EMZ03], for a sufficiently small neighborhood V of z and sufficiently small ϵ the map φ defines a homeomorphism of $V \times B(\epsilon)$ onto a neighborhood U of z in $\overline{\mathcal{Q}}$ with the properties stated in the lemma. \square

The measure in the statement of the following lemma is the Lebesgue measure.

Lemma 2.6. *For almost every $\omega \in \mathcal{Q}$, the leaf $\mathcal{AP}(\omega)$ intersects every infinite core face of the principal boundary of \mathcal{Q} in regular points.*

Proof. We say that a translation surface ω has an *isolated bigon* of type (g_1, g_2) where $g_1 + g_2 = g$ if it admits a pair of homologous saddle connections α_1, α_2 connecting two zeros p_1, p_2 with the following property. There is no other saddle connection parallel to α_i , moreover $\alpha_1 \cup \alpha_2$ decomposes S into a surface of genus g_1 and a surface of genus g_2 . Since the Teichmüller flow preserves saddle connections and only changes their direction, the set of points $q \in \mathcal{Q}$ which admit an isolated bigon of type (g_1, g_2) is invariant under the Teichmüller flow.

The set of directions of a translation surface containing a saddle connection is countable and hence of measure zero. Thus Proposition 2.5 shows that for all $g_1, g_2 \geq 1$ with $g_1 + g_2 = g$, the set of all points $\omega \in \mathcal{Q}$ which admit an isolated bigon of type (g_1, g_2) has positive Lebesgue measure (see also [EMZ03] for details). By invariance and by ergodicity of the Teichmüller flow on \mathcal{Q} , we conclude that this set has full measure.

Let $\omega \in \mathcal{Q}$ and assume that there is some $e^{is} \in S^1$ with the property that $e^{is}\omega$ has an isolated horizontal bigon. Assume that this bigon is defined by a pair of homologous saddle connections with endpoints at the zeros p_1, p_2 of ω . We assume that the points p_1, p_2 are ordered in such a way that the saddle connections connect p_1 to p_2 with respect to the orientation defined by ω . Then the flow line $t \rightarrow \Lambda_{\mathbf{a}}^t(e^{is}\omega)$ defined by the vector \mathbf{a} with coordinates $(-1, 1)$ at the points p_1, p_2 collapses the pair of horizontal saddle connections of $e^{is}\omega$ to a point. This means that there is some $\tau > 0$ such that $\Lambda_{\mathbf{a}}^t(e^{is}\omega)$ is defined for $0 \leq t < \tau$, and the surfaces $\Lambda_{\mathbf{a}}^t(e^{is}\omega)$ converge as $t \nearrow \tau$ to a surface in the infinite core face of the principal boundary of type (g_1, g_2) . The lemma follows. \square

A subset of \mathcal{Q} is *saturated for the absolute period foliation* if it is a union of leaves.

Corollary 2.7. *The set \mathcal{S} of points $\omega \in \mathcal{Q}$ such that $\mathcal{AP}(\omega)$ intersects every infinite core face of the principal boundary of \mathcal{Q} is saturated for the absolute period foliation and of full Lebesgue measure.*

A smooth foliation of \mathcal{Q} is *ergodic* for the Lebesgue measure if every Borel set $A \subset \mathcal{Q}$ which is saturated for the foliation either has full measure or measure zero.

A finite core face of the principal boundary of a stratum is a component of a stratum. Hence if $g \geq 3$ and if this stratum has more than one zero, then the absolute period foliation is defined. For an infinite core face of the principal boundary, the absolute period foliation is defined as well. Namely, such an infinite core face is determined by closed surfaces S_1, S_2 of genus $g_1 \geq 1, g_2 \geq 1$ and $g_1 + g_2 = g$. Write $S_1 \sqcup S_2$ for the surface obtained by attaching S_1 and S_2 at a single point, viewed as a surface with a node. The moduli spaces of abelian differentials on S_1, S_2 determine a moduli space of abelian differentials on $S_1 \sqcup S_2$ and an absolute period foliation. We require that the area of a differential on $S_1 \sqcup S_2$ equals one and hence the areas of S_1 and S_2 add up to one. In particular, the principal stratum in the moduli space of abelian differentials on $S_1 \sqcup S_2$ decomposes as $\cup_{a \in (0,1)} \mathcal{Q}_{S_1 \sqcup S_2}(a, 1-a)$ where a differential in $\mathcal{Q}_{S_1 \sqcup S_2}(a, 1-a)$ gives area a to S_1 . Note that for each $a \in (0, 1)$, $\mathcal{Q}_{S_1 \sqcup S_2}(a, 1-a)$ is a real hypersurface in the moduli space of all area one abelian differentials on $S_1 \sqcup S_2$.

Lemma 2.8. *If the absolute period foliation of the principal stratum of S_1, S_2 is ergodic then so is the absolute period foliation of $\mathcal{Q}_{S_1 \sqcup S_2}(a, 1-a)$.*

Proof. Write $a_1 = a$ and $a_2 = 1 - a$. Then $\mathcal{Q}_{S_1 \sqcup S_2}(a_1, a_2)$ is the space of pairs $((\omega_1, p_1), (\omega_2, p_2))$ where ω_i is an abelian differential on S_i of area a_i with simple zeros and a marked point p_i (here the node is at the marked point).

Let $\mathcal{Q}_{S_i}(a_i)$ be the principal stratum in the moduli space of abelian differentials on S_i of area a_i . There is a natural node forgetting map

$$P : \mathcal{Q}_{S_1 \sqcup S_2}(a_1, a_2) \rightarrow \mathcal{Q}_{S_1}(a_1) \times \mathcal{Q}_{S_2}(a_2).$$

This map is a fibration whose fibre over a point (ω_1, ω_2) can naturally be identified with the product $(S_1, \omega_1) \times (S_2, \omega_2)$ (it consists of the pair of marked points) and hence it is equipped with a natural Lebesgue measure (the product of the Lebesgue measures defined by the differentials ω_i on the surfaces S_i). The fibration respects absolute periods and therefore if $\omega \in \mathcal{Q}_{S_1 \sqcup S_2}(a_1, a_2)$ then $P^{-1}(P\omega) \in \mathcal{AP}(\omega)$.

The Lebesgue measure on $\mathcal{Q}_{S_1 \sqcup S_2}(a_1, a_2)$ can locally be described as a product of the Lebesgue measure on the fibre and the Lebesgue measure on the base (see [EMZ03] for details). As a consequence, a Borel set $A \subset \mathcal{Q}_{S_1 \sqcup S_2}(a_1, a_2)$ which is saturated for the absolute period foliation maps to a Borel subset of $\mathcal{Q}_{S_1}(a_1) \times \mathcal{Q}_{S_2}(a_2)$ which is saturated for the absolute period foliation, and it coincides with $P^{-1}(PA)$ up to a set of measure zero. Thus ergodicity of the absolute period foliation on $\mathcal{Q}_{S_i}(a_i)$ implies ergodicity of the absolute period foliation on $\mathcal{Q}_{S_1 \sqcup S_2}(a_1, a_2)$. \square

As an immediate consequence we obtain

Corollary 2.9. *Let $A \subset \mathcal{Q}$ be a Borel set which is saturated for the absolute period foliation and of positive Lebesgue measure. Let $U = V \times B(\epsilon)$ be a standard neighborhood in an infinite core face \mathcal{F} defined by a surface with nodes $S_1 \sqcup S_2$. If the absolute period foliation of the principal stratum of S_1, S_2 is ergodic then there is a Borel set $C \subset (0, 1)$ of positive Lebesgue measure such that up to a set of measure zero we have*

$$A \cap U = (\cup_{s \in C} \mathcal{Q}_{S_1 \sqcup S_2}(s, 1-s) \cap V) \times B(\epsilon).$$

Proof. By Proposition 2.5, up to a set of measure zero the set A intersects U in a set of the form $(Z \cap V) \times B(\epsilon)$ where $Z \subset \mathcal{F}$ is a Borel set which is saturated for the absolute period foliation. If the absolute period foliation of S_1, S_2 is ergodic then Lemma 2.8 shows that there is a Borel set $C \subset (0, 1)$ of positive measure such that $Z = \cup_{s \in C} \mathcal{Q}_{S_1 \sqcup S_2}(s, 1-s)$ as claimed. \square

We use Lemma 2.8 to show

Proposition 2.10. *The absolute period foliation $\mathcal{AP}(\mathcal{Q})$ of the principal stratum in genus $g \geq 2$ is ergodic.*

Proof. We use induction on the genus g of S . The case $g = 2, 3$ is due to McMullen [McM14]. Let $g \geq 6$ and assume that the proposition holds true for $g - 4$ and for $g - 2$.

Let \mathcal{Q} be the principal stratum of abelian differentials on a surface of genus g . Let $A \subset \mathcal{Q}$ be a Borel subset which is saturated for the absolute period foliation and which is of positive Lebesgue measure. Then the same holds true for $A \cap \mathcal{S}$ where $\mathcal{S} \subset \mathcal{Q}$ is as in Corollary 2.7.

Let S_1, S_3 be a surface of genus two and let $S_1 \sqcup S_2 \sqcup S_3$ be the surface with two nodes obtained by attaching S_1, S_3 to a surface S_2 of genus $g - 4$ at a single point

each. Lift \mathcal{Q} to a finite cover $\hat{\mathcal{Q}}$ so that this configuration determines a subset of $\hat{\mathcal{Q}}$ as follows. Let \mathcal{Z} be the space of all area one abelian differentials on $S_1 \sqcup S_2 \sqcup S_3$. Note that

$$\mathcal{Z} = \bigcup_{a_i > 0, a_1 + a_2 + a_3 = 1} \mathcal{Z}(a_1, a_2, a_3)$$

where an abelian differential in the space $\mathcal{Z}(a_1, a_2, a_3)$ gives area a_i to S_i . We require that there is an open subset U of $\hat{\mathcal{Q}}$ of the form

$$U = V \times B(\epsilon) \times B(\epsilon)$$

with $V \subset \mathcal{Z}$ open and such that for each $x \in V$ the set $\{x\} \times B(\epsilon) \times B(\epsilon)$ is contained in a leaf of the absolute period foliation. The existence of such an open set U follows from two applications of Proposition 2.5 and the requirement that we can distinguish the two surfaces S_1, S_3 in the principal boundary of $\hat{\mathcal{Q}}$ (i.e. they are not identified by an element of the mapping class group).

As before, we conclude that since A is saturated for the absolute period foliation, there is a Borel subset \mathcal{Z}_A of \mathcal{Z} which is saturated for the absolute period foliation and such that up to a set of measure zero, the intersection of A with $V \times B(\epsilon) \times B(\epsilon)$ equals

$$(V \cap \mathcal{Z}_A) \times B(\epsilon) \times B(\epsilon) \cap \hat{\mathcal{Q}}.$$

By induction hypothesis, the absolute period foliation on S_i is ergodic. Note that as we assume that $g \geq 6$, the genus of S_2 is at least two. By Lemma 2.8, this implies that there is a Borel subset D_0 of the set

$$D = \{(a_1, a_2, a_3) \mid a_i > 0, a_1 + a_2 + a_3 = 1\}$$

such that up to a set of measure zero, we have $\mathcal{Z}_A = \bigcup_{x \in D_0} \hat{\mathcal{Q}}_{S_1 \sqcup S_2 \sqcup S_3}(x)$. Moreover, the Lebesgue measure of D_0 is positive.

Let Σ be a surface of genus $g - 2$ (which should be viewed as a component of a surface with a single node in the Deligne Mumford compactification of the moduli space of S whose second component is the surface S_1). For each $a < 1$, the surface with nodes $S_2 \sqcup S_3$ determines an infinite core face $\mathcal{G}(1 - a)$ of the moduli space $\mathcal{H}_\Sigma(1 - a)$ of abelian differentials on Σ of area $1 - a$. By Corollary 2.9 and the induction hypothesis, applied to S_1 and Σ , there is a Borel set $C_1 \subset (0, 1)$ such that

$$A \cap U = \left(\bigcup_{a \in C_1} (\hat{\mathcal{Q}}_{S_1 \sqcup \Sigma}(a, 1 - a) \cap V \times B(\epsilon)) \times B(\epsilon) \right) \cap \hat{\mathcal{Q}}.$$

As a consequence, the set D_0 is of the form

$$(1) \quad D_0 = \{(a_1, a_2, a_3) \mid a_1 \in C_1, a_1 + a_2 + a_3 = 1\}.$$

Exchanging the roles of S_1 and S_3 shows that on the other hand, there is a Borel set $C_3 \subset (0, 1)$ such that

$$(2) \quad B_0 = \{(a_1, a_2, a_3) \mid a_3 \in C_3, a_1 + a_2 + a_3 = 1\}.$$

Let λ be the Lebesgue measure on $(0, 1)$. Define $\text{essup}(C_i) = \sup\{a > 0 \mid \lambda(C_i \cap [a, 1]) > 0\} \in (0, 1]$ and $\text{essinf}(C_i) = \inf\{a > 0 \mid \lambda(C_i \cap [0, a]) > 0\} \in [0, 1)$ ($i = 1, 3$). It follows from equation (1) and equation (2) that λ -almost every $b < 1 - \text{essinf}(C_1)$ is contained in C_3 (since on the one hand, we can make a_2 as small as we wish, on the other hand, for fixed $a \in C_1$ we can make a_2 as close to $1 - a$ as we wish). As a consequence, the set C_3 is of the form $(0, c)$ for a number $c > 0$, in

particular we have $\text{essinf}(C_3) = 0$. By symmetry, we conclude that $\text{essinf}(C_1) = 0$ as well and hence by the beginning of this paragraph, $C_3 = (0, 1) = C_1$. As a consequence, $D_0 = D$ which shows that A is of full Lebesgue measure. Ergodicity of the absolute period foliation is an immediate consequence.

The proposition follows by induction if we can show ergodicity of the absolute period foliation for $g = 4$ and $g = 5$.

We begin with the case $g = 4$. To this end consider a core face of the principal boundary defined by a surface with nodes $S_1 \sqcup S_2$ where S_i is a surface of genus 2.

Let $A \subset \mathcal{Q}$ be a Borel set saturated for the absolute period foliation and of positive Lebesgue measure. We use a neighborhood in standard form of a core face of the principal boundary to deduce that there is a Borel set $C_1 \subset (0, 1)$ so that

$$A \cap U = (\cup_{a \in C_1} \hat{\mathcal{Q}}_{S_1 \sqcup S_2}(a, 1 - a) \cap V) \times B(\epsilon).$$

The principal boundary of the surface S_1 has an infinite core face which consists in surfaces with a separating node given by a decomposition of S_1 into two tori T_1, T_2 . The node is a regular point on each torus. The absolute period foliation is the foliation defined by moving the marked point over the torus. Thus as before, for a fixed number $s > 0$, there is a forgetful projection $P_{\mathcal{T}}$ of the moduli space $\mathcal{T}(s)$ of tori of area s with a marked point onto the moduli space of tori of area s without marked point. A Borel set in $\mathcal{T}(s)$ which is saturated for the absolute period foliation is the preimage under $P_{\mathcal{T}}$ of a Borel set $B(s)$ of the moduli space of tori of area s .

As a consequence, if we denote again by

$$P : \cup_{s \in (0, 1)} \cup_s \mathcal{Q}_{T_1 \sqcup \Sigma}(s, 1 - s) \rightarrow \mathcal{Q}_{T_1}(s) \times \mathcal{Q}_{\Sigma}(1 - s)$$

the natural projection where Σ is of genus 3, then the set A intersects a neighborhood of the core face defined by $T_1 \sqcup \Sigma$ in a set of the form

$$\cup_s P^{-1}B(s) \times \mathcal{Q}_{\Sigma}(s).$$

However, using ergodicity of the absolute period foliation for surfaces of genus 2, we know that for each s , either $B(s)$ is of full measure or of measure zero. By this observation, we can use the above argument in the case $g = 4$ as well.

The proof for $g = 5$ is completely analogous and will be omitted. \square

3. THE PRINCIPAL BOUNDARY OF AFFINE INVARIANT MANIFOLDS

The goal of this section is to study the intersection of an *affine invariant submanifold* of a stratum \mathcal{Q} with $k \geq 2$ zeros with a leaf of the absolute period foliation.

To this end call a connected subset B of a leaf of $\mathcal{AP}(\mathcal{Q})$ *complex affine* if each point $p \in B$ has an open neighborhood U which in affine coordinates is an open subset of a complex affine subspace. This is equivalent to stating that B is a smooth submanifold of a leaf of $\mathcal{AP}(\mathcal{Q})$ whose tangent bundle TB is invariant under the complex structure and the real structure and whose lift to $\hat{\mathcal{Q}}$ is invariant under all

flows $\Lambda_{e^{is}\mathbf{a}}^t$ whenever $e^{is}X_{\mathbf{a}} \in TB$. Here as before, $\hat{\mathcal{Q}}$ is a finite cover of \mathcal{Q} so that the zeros of differentials in $\hat{\mathcal{Q}}$ are numbered. Moreover, for any vector $\mathbf{a} \in \mathbb{R}^k$ of zero mean, $X_{\mathbf{a}}$ is the Schiffer variation defined by \mathbf{a} .

The *rank* of an affine invariant manifold \mathcal{C} is defined by

$$\text{rk}(\mathcal{C}) = \frac{1}{2}\dim(pTC)$$

where p is the projection of period coordinates into absolute cohomology.

Lemma 3.1. *Let $\mathcal{C} \subset \mathcal{Q}$ be an affine invariant submanifold. Then for every $\omega \in \mathcal{C}$, the intersection $\mathcal{C} \cap \mathcal{AP}(\omega)$ is a complex affine subspace of complex dimension $\dim(\mathcal{C}) - 2\text{rk}(\mathcal{C})$.*

Proof. A proper affine invariant submanifold of \mathcal{Q} lifts to a proper affine invariant submanifold of $\hat{\mathcal{Q}}$, so it suffices to consider such manifolds \mathcal{C} in $\hat{\mathcal{Q}}$. Let $r = \dim(\mathcal{C}) - 2\text{rk}(\mathcal{C})$. We may assume that $r > 0$. Then for each $q \in \mathcal{C}$ there is a vector $X \in T_q\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to \mathcal{C} . By invariance of \mathcal{C} under the Teichmüller flow, we have $d\Phi^t(X) \in T\mathcal{AP}(\hat{\mathcal{Q}}) \cap TC$ for all t .

A vector $X \in T\mathcal{AP}(\hat{\mathcal{Q}}) \cap TC$ decomposes as $X = X^u + X^s$ where $X^u \in T\mathcal{AP}(\hat{\mathcal{Q}})$ is real (and hence tangent to the strong unstable foliation) and X^s is imaginary (and hence tangent to the strong stable foliation). We claim that we can find a vector $Y \in TC \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which either is tangent to the strong unstable or to the strong stable foliation. To this end we may assume that $X^u \neq 0$. Since this is an open condition and since the Teichmüller flow on \mathcal{C} is topologically transitive, we may furthermore assume that the Φ^t -orbit of the footpoint q of X is dense in \mathcal{C} . Then there is a sequence $t_i \rightarrow \infty$ such that $\Phi^{t_i}(q) \rightarrow q$.

Choose any smooth norm $\|\cdot\|$ on $T\hat{\mathcal{Q}}$. Up to passing to a subsequence,

$$d\Phi^{t_i}(X)/\|d\Phi^{t_i}(X)\|$$

converges to a vector $Y \in T_q\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to the strong unstable foliation. As the bundle $TC \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ is a smooth $d\Phi^t$ -invariant subbundle of the restriction of the tangent bundle of $\hat{\mathcal{Q}}$ to \mathcal{C} , we have $Y \in TC \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which is what we wanted to show.

Using Lemma 2.2 and density of the Φ^t -orbit of q , if $0 \neq \mathbf{a} \in \mathbb{R}^k$ is such that $Y = X_{\mathbf{a}}(q)$ then $X_{\mathbf{a}}(u) \in TC$ for all $u \in \mathcal{C}$. As a consequence, \mathcal{C} is invariant under the flow $\Lambda_{\mathbf{a}}^t$.

By invariance of TC under the complex structure J , if $r = 1$ then

$$TC \cap T\mathcal{AP}(\hat{\mathcal{Q}}) = \mathbb{R}X_{\mathbf{a}} \oplus J\mathbb{R}X_{\mathbf{a}}$$

and we are done. Otherwise there is a tangent vector $X \in TC \cap T\mathcal{AP}(\hat{\mathcal{Q}}) - \mathbb{C}X_{\mathbf{a}}$. Apply the above argument to X , perhaps via replacing the Teichmüller flow by its inverse. In finitely many such steps we conclude that there is a smooth subbundle \mathcal{R} of $TC \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to the strong unstable foliation (i.e. real for the real structure) and of rank r such that $TC \cap T\mathcal{AP}(\hat{\mathcal{Q}}) = \mathbb{C}\mathcal{R}$. Moreover, if $\omega \in \mathcal{C}$ and if $X_{\mathbf{a}}(\omega) \in \mathcal{R}$ then $X_{\mathbf{a}}(q) \in \mathcal{R}$ for every $q \in \mathcal{C}$ and \mathcal{C} is invariant under

the flow $\Lambda_{\mathbf{a}}^t$. The same argument applied to the imaginary subbundle $i\mathcal{R}$ of TC and equivariance under the action of the circle group of rotations yields the statement of the lemma. \square

We now always assume that $\mathcal{C} \subset \mathcal{Q}$ is an affine invariant manifold such that the (complex) dimension of $TC \cap T\mathcal{AP}(\mathcal{Q})$ is at least one. We call such an affine invariant manifold *redundant*. By Lemma 3.1, there are Schiffer variations $X_{\mathbf{a}} \subset T\mathcal{AP}(\hat{\mathcal{Q}})$ which are tangent to \mathcal{C} .

Let $q \in \mathcal{Q}$ and let α be a horizontal saddle connection for q of length $\ell(\alpha)$; here the length is taken with respect to the flat metric. For a vector $\mathbf{a} \in \mathbb{R}^k$ with zero mean define the *oriented \mathbf{a} -weighted length* of α by

$$w_{\mathbf{a}}(\alpha) = \ell(\alpha)/b$$

where b is the oriented difference of the coordinates of \mathbf{a} at the endpoints of α (i.e. the weight of the incoming endpoint minus the weight of the outgoing endpoint for the orientation defined by ω) provided that this difference does not vanish, and define $w_{\mathbf{a}}(\alpha) = \infty$ otherwise. Note that if we replace \mathbf{a} by any nonzero real multiple then the oriented \mathbf{a} -weighted length of a saddle connection multiplies with the inverse of the multiple. As a consequence, the set of horizontal saddle connections whose oriented \mathbf{a} -weighted length is positive and minimal is invariant under scaling $X_{\mathbf{a}}$ with a positive number.

An *oriented cycle* of horizontal saddle connections for an abelian differential ω is an embedded closed curve in S consisting of at least two horizontal saddle connections, and these saddle connections are equipped with the orientation induced by ω . We do not require that these orientations fit together to an orientation of the simple closed curve. An example of an oriented cycle is a *bigon* which consists of two distinct homologous horizontal saddle connections of the same length which meet at two distinct endpoints with an angle an integral multiple of 2π . Cycles of combinatorial length u pass through u distinct zeros of ω .

Remark 3.2. As explained on p.73 of [EMZ03], a translation surface which is generic for the Lebesgue measure in a stratum has infinitely many bigons of saddle connections- in fact, the number of these bigons grows in length with the same rate as the number of all saddle connections.

Definition 3.3. Let $\mathcal{C} \subset \hat{\mathcal{Q}}$ be a redundant affine invariant manifold and let $X_{\mathbf{a}} \in T\mathcal{AP}(\hat{\mathcal{Q}})$ be tangent to \mathcal{C} . Define $q \in \mathcal{C}$ to be *flexible* for $X_{\mathbf{a}}$ if there is some $s \in [0, 2\pi)$ with the following properties.

- (1) The graph G of horizontal saddle connections for $e^{is}q$ is nonempty and contains elements of finite \mathbf{a} -weighted length.
- (2) The subgraph of G of saddle connections for $e^{is}q$ with smallest positive \mathbf{a} -weighted length does not contain cycles.

A translation surface which is not flexible for $X_{\mathbf{a}}$ is called *rigid* for $X_{\mathbf{a}}$. We call the vector field $X_{\mathbf{a}}$ *flexible* for \mathcal{C} if there is a flexible translation surface $q \in \mathcal{C}$ for $X_{\mathbf{a}}$.

Note that if $q \in \mathcal{C}$ is flexible for $X_{\mathbf{a}}$ then q is flexible for $cX_{\mathbf{a}}$ for every $c \in \mathbb{R} - \{0\}$ (to see this for a negative number c use the fact that $-\omega = e^{\pi i} \omega \in \mathcal{C}$). In particular, if we denote by \mathcal{V} the real subspace of the complex vector space $T\mathcal{C} \cap T\mathcal{AP}(\mathcal{Q})$ then we can talk about flexible points in the projectivization $P\mathcal{V} \subset P\mathbb{R}^k$ of \mathcal{V} .

The next lemma states the basic properties of flexible vector fields.

- Lemma 3.4.** (1) *If $[X_{\mathbf{a}}] \in P\mathcal{V}$ is flexible for $q \in \mathcal{C}$ then there are open neighborhoods V of $[X_{\mathbf{a}}]$ in $P\mathcal{V}$, U of q in \mathcal{C} such that every $[Y] \in V$ is flexible for every $z \in U$.*
- (2) *For every $[X_{\mathbf{a}}] \in P\mathcal{V}$ the set of all $[X_{\mathbf{a}}]$ -rigid points in \mathcal{C} is a finite (perhaps empty) union of affine invariant manifolds which is proper if and only if $[X_{\mathbf{a}}]$ is flexible for \mathcal{C} .*

Proof. Let $\mathbf{a} \in \mathbb{R}^k$ and assume that $q \in \mathcal{C}$ is flexible for $X_{\mathbf{a}}$. Let e^{is} be such that the graph G of horizontal saddle connections of $e^{is}q$ of minimal positive \mathbf{a} -weighted length is not empty and does not have cycles. Recall that G is composed of finitely many saddle connections.

Let $\alpha_1, \dots, \alpha_k$ be the saddle connections of minimal positive \mathbf{a} -weighted length. Since saddle connections for differentials q depend smoothly on q as long as q remains in a fixed component of a stratum, for each $z \in \mathcal{C}$ which is sufficiently close to $e^{is}q$ and for each i there is a saddle connection $\alpha_i(z)$ for z which is homotopic to α_i with fixed endpoints (for the natural identification of nearby surfaces) and which depends smoothly on z . Moreover, the set of saddle connections in the direction of $\alpha_i(z)$ of length bounded from above by some fixed number can be obtained by a smooth deformation of some (perhaps not all) saddle connections of q .

As a consequence, for each point z in a neighborhood W of q there is some i such that $\alpha_i(z)$ is of minimal positive \mathbf{a} -weighted length in its direction. Moreover, since the graph G has no cycles there are no cycles of saddle connections of minimal positive \mathbf{a} -weighted length on $z \in W$ whose direction coincides with the direction of $\alpha_i(z)$. This is what we wanted to show.

To summarize, the set of points which are flexible for $X_{\mathbf{a}}$ is open. The same reasoning shows that a point which is flexible for $X_{\mathbf{a}}$ is flexible for $X_{\mathbf{b}}$ for every $\mathbf{b} \in \mathbb{R}^k$ sufficiently close to \mathbf{a} . The first part of the lemma follows.

To show the second part of the lemma, by Theorem 2.2 of [EMM13] it suffices to show that the set of points in \mathcal{C} which are flexible for $X_{\mathbf{a}}$ is invariant under the action of the group $SL(2, \mathbb{R})$. Now the group $SL(2, \mathbb{R})$ is generated by the circle group of rotations and the group of diagonal matrices and therefore it suffices to show that the set of all flexible points for $X_{\mathbf{a}}$ in \mathcal{C} is invariant under the action of these two groups.

Invariance under the action of the circle group of rotations is immediate from the definition. To show invariance under the action of the diagonal group note that this action preserves saddle connections and maps saddle connections with the same slope to saddle connections with the same slope. Moreover, lengths of saddle connections with the same slope are multiplied with the same constant. The claim then follows from Lemma 2.2. \square

Example 3.5. 1) There is an affine invariant manifold \mathcal{C} of rank one and dimension 3 in the principal stratum $\mathcal{H}(1, 1)$ of the moduli space of abelian differentials on a surface of genus $g = 2$. This manifold is mapped by the composition of the projection $\mathcal{H}(1, 1) \rightarrow \mathcal{M}_2$ (here \mathcal{M}_2 is the moduli space of curves of genus 2) with the Torelli map into the Hilbert modular surface $\mathbf{H}^2 \times \mathbf{H}^2 / SL(2, \mathcal{O}_{\sqrt{5}})$ for discriminant $D = 5$. The principal boundary of \mathcal{C} intersects the finite face $\mathcal{H}(2)$ of $\mathcal{H}(1, 1)$ in a Teichmüller curve. The rigid set for the (unique up to scale) real vector field X_a consists of another Teichmüller curve, the curve through the translation surface defined by the regular decagon. For no other discriminant such a rigid curve in the corresponding rank one affine invariant manifold exists [McM05, McM06].

2) Let \mathcal{B} be the moduli space of holomorphic differentials on \mathbb{CP}^1 with 12 simple poles and 4 double zeros. Taking a two-sheeted branched cover with a branch point at each of the singular points defines an $SL(2, \mathbb{R})$ -invariant closed subset \mathcal{C} of the stratum \mathcal{Q} of abelian differentials with four double zeros on a surface of genus 5. The hyperelliptic involution acts on saddle connections connecting the zeros and hence saddle connections in a given direction come in pairs. As a consequence, \mathcal{C} is contained in the rigid set of a flexible vector field.

As in the case of strata, we can talk about the principal boundary of an affine invariant manifold and its faces. The goal of the following proposition is to investigate the finite faces of the principal boundary of a redundant affine invariant manifold $\mathcal{C} \subset \mathcal{Q}$. If $k \geq 2$ is the number of zeros of \mathcal{Q} then these finite faces are contained in strata whose number of zeros is smaller than $k - 1$.

By compatibility of the $SL(2, \mathbb{R})$ -action with closures of strata, the closure of an affine invariant manifold $\mathcal{C} \subset \mathcal{Q}$ in the entire moduli space \mathcal{H} of abelian differentials is a closed $SL(2, \mathbb{R})$ -invariant set and hence a finite union of affine invariant manifolds [EMM13]. Note that this closure may just be \mathcal{C} , e.g. when \mathcal{C} is a Teichmüller curve.

If this closure does *not* coincide with \mathcal{C} then we can talk about the finite principal boundary of \mathcal{C} and its faces as before. It is a finite union of affine invariant manifolds. If $\mathcal{B} \subset \overline{\mathcal{C}}$ is such a finite face then we say that $\mathcal{B} \subset \overline{\mathcal{C}}$ is *embedded in standard form* if every $q \in \mathcal{B}$ has a neighborhood V with the following property. There exists a number $\epsilon > 0$ and a homeomorphism $\varphi : V \times B(\epsilon) \rightarrow \overline{\mathcal{C}}$ onto a neighborhood of q in $\overline{\mathcal{C}}$ such that $\varphi(z, 0) = z$ and $\varphi(\{z\} \times B(\epsilon)) \subset \mathcal{AP}(z)$ for all $z \in V$.

In the next proposition, the number of zeros of the component of $\overline{\mathcal{C}} - \mathcal{C}$ may be strictly smaller than $k - 1$.

Proposition 3.6. *Let $\mathcal{C} \subset \mathcal{Q}$ be an affine invariant manifold of rank $\ell \geq 1$ and dimension $2\ell + r$ for some $r > 0$ and let $\overline{\mathcal{C}}$ be the closure of \mathcal{C} in \mathcal{H} . If \mathcal{C} is flexible then $\overline{\mathcal{C}} - \mathcal{C}$ contains a nonempty finite union of affine invariant manifolds of rank ℓ and dimension $2\ell + r - 1$ which are embedded in $\overline{\mathcal{C}}$ in standard form.*

Proof. As before, we pass to the cover $\hat{\mathcal{Q}}$ with numbered zeros. Thus let $\mathcal{C} \subset \hat{\mathcal{Q}}$ be an affine invariant manifold of rank $\ell \geq 1$ and dimension $2\ell + r$ for some $r > 0$. By Lemma 3.1, \mathcal{C} intersects each leaf of the absolute period foliation in an affine subspace. In particular, there is a linear subspace \mathcal{V} of \mathbb{R}^k of points of zero mean

so that for each $\mathbf{a} \in \mathcal{V}$ and every $q \in \mathcal{C}$, the flow line $\Lambda_{\mathbf{a}}^t q$ is contained in \mathcal{C} as long as it is defined.

Assume that there is some $q \in \mathcal{C}$ and some $\mathbf{a} \in \mathcal{V}$ such that q is flexible for $X_{\mathbf{a}}$. By definition, there is some $s \in [0, 2\pi)$ such that the differential $e^{is}q$ admits a finite graph G of saddle connections of minimal positive oriented \mathbf{a} -weighted length, and this graph does not contain any cycles. As in the proof of Lemma 3.4, we may assume that this graph depends smoothly on q in the following sense. There is a local smooth transversal $\mathcal{V} \subset \mathcal{C}$ to the circle action and a smooth function $\sigma : \mathcal{V} \rightarrow \mathbb{R}$ through $\sigma(q) = s$ so that for all $z \in \mathcal{V}$ there is a graph $G(z)$ of saddle connections on z which are horizontal for $e^{i\sigma(z)}z$ so that $G(z)$ is of minimal \mathbf{a} -weighted length and depends smoothly on z .

Let $\alpha \subset G(z)$ be such a saddle connection of minimal \mathbf{a} -weighted length. Then the length of α decreases under the Schiffer variation through $e^{i\sigma(z)}z$ which is defined by $X_{\mathbf{a}}$. We claim that the flow $\Lambda_{\mathbf{a}}^t$ of $X_{\mathbf{a}}$ through $e^{i\sigma(z)}z$ limits on a surface in $\overline{\mathcal{C}}$ which does not have nodes. Namely, by assumption $G(z)$ is a finite union of trees and hence by the definition of the oriented \mathbf{a} -weighted length and the properties of the Schiffer variations discussed in Section 2, the limiting surface is obtained by collapsing each of these trees to a point and hence identifying the vertices of each of these trees. Hence if $G(z)$ is connected and has $u \geq 1$ edges then the limiting surface is contained in the stratum $\mathcal{Q}_1 \subset \overline{\mathcal{Q}}$ of differentials with $k - u$ zeros.

As z varies through \mathcal{V} , the limiting surfaces define a subset V of a component of a stratum with $k - u$ zeros which is contained in the closure of \mathcal{C} . This set is transverse to the action of the unit circle by multiplication with a complex number of absolute value one. The map can be extended to an S^1 -equivariant map of a neighborhood of \mathcal{V} onto a neighborhood of the image. The complex dimension of the image equals the complex dimension of \mathcal{C} minus one.

As a consequence, the closure of \mathcal{C} intersects a boundary stratum of \mathcal{Q} of codimension at least one in a set which is of dimension at least $\dim(\mathcal{C}) - 1$. The statement of the proposition now follows from invariance of the closure of \mathcal{C} under the action of $SL(2, \mathbb{R})$. \square

Conjecture: A rank ℓ submanifold of a rank $\ell \geq 2$ affine invariant manifold \mathcal{C} is contained in the rigid set of a flexible vector field for \mathcal{C} .

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